Ultimate Nim:
The Use of Nimbers, Binary Numbers and Subpiles in the Optimal Strategy for Nim

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1.0 THE ORIGIN AND TYPES OF NIM

1.1 *Origin.* Nim was given its current name by the American mathematician Charles L. Bouton, from an Old English word meaning "take"; however, the modern version of Nim resembles an ancient Chinese game called Tsyanshidzi (or "picking stones"). (Tsyanshidzi most closely resembles Wythoff's Nim.) The rules for Nim are simple—the game is played by two people with a number of "tokens"[1] arranged in two or more rows (known as "nim heaps"[2]). Each player takes a turn removing one or more tokens from a single nim heap. The object of the game varies, depending on the version; several versions are described below.

1.2 *Common Nim.* In "common" Nim, a player's objective is to force his opponent to take the last token (a misère-form game). For example, in a simplified position, with one token in the first nim heap and two tokens in the second heap (a 1-2 position), the first player could win the game by removing both tokens from the second heap, forcing the second player to remove the lone token from the first heap. The optimal strategies for Nim discussed in section 3 must be augmented for common Nim by the end game discussed in section 4.

1.3 *Straight Nim.* In "straight" Nim, a player's objective is to take the last token himself. For example, in the 1-2 position described above, the first player could win the game by removing one of the tokens from the second heap, forcing the second player to take either the one token from the first heap or the remaining token from the second heap, leaving one token for the first player to take for the win.

1.4 *Single-pile Nim.* Single-pile Nim may be played as "common" or "straight," but all tokens are in a single nim heap, and each player takes between one and n (for example, five) tokens in each turn.

1.5 *Nim_k (Moore's Nim).* Nim_k, or "Moore's Nim," may be played as "common" or "straight," but each player may take one or more tokens from a number of nim heaps less than or equal to k. Thus, the examples described above, in the terminology of Moore's Nim, were played as Nim_1, where each player could only take from a single heap in their turn.

1.6 *Wythoff's Nim.* "Wythoff's Nim" is a variant of straight Nim where there are only two heaps, and each player may remove tokens from one or both heaps, but if a player removes from both heaps, he must remove the same number of tokens from each.

1.7 *Marienbad.* Marienbad is a form of common Nim with a 1-3-5-7 starting position featured in Alain Resnais' 1961 film, *L'année dernière à Marienbad* (*Last Year at Marienbad*), where playing cards and matchsticks were used as tokens. Because the starting position is balanced, the first player is at a disadvantage. In the film, the character who introduces the game graciously allows his opponent to take the first turn.

2.0 BASIC NIM STRATEGY

Nim strategies are used to determine whether a given position is balanced or unbalanced, and what moves will balance the position for the opponent's turn. A balanced position[3], when modified by a player's turn, will

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[1] *The term "token" is used herein as a generic term to refer to whatever item is used in play (e.g., cards, coins, matchsticks, marbles, etc.).*

[2] *The term "nim heap" is used herein as a generic term to refer to a group of tokens. A nim heap may be a row, column, pile, etc., depending on the form of play.*

[3] *A balanced position is also referred to in Nim literature as an "even position," an "unsafe position," or a "losing position."*

always be unbalanced. An unbalanced position[4], when modified by a player's turn, may be balanced or remain unbalanced. The flow of a Nim game, in balanced and unbalanced positions, is represented in figure 1.

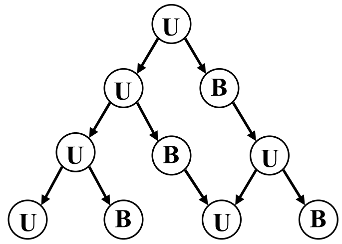

figure 1. Nim flow tree

A player who is offered a balanced position is at a disadvantage; therefore, the objective in each turn is to offer your opponent a balanced position, which they have no choice but to unbalance.

However, if the starting position is balanced, or your opponent takes first and offers you a balanced position, you must count on your opponent to make an error (and leave you an unbalanced position) for a chance to employ a Nim strategy for a win.

3.0 OPTIMAL STRATEGIES FOR NIM

3.1 *Subpiles.* The subpile optimal strategy for Nim consists of evaluating whether or not a position is balanced by breaking down each nim heap into subpiles, with each subpile containing an exact power of two. Thus, a heap with nine tokens is broken into a subpile with eight tokens (2^3) and a subpile with one token (2^0), a heap with six tokens is broken into a subpile with four tokens (2^2) and a subpile with two tokens (2^1), etc. In each case, a heap is broken into the minimum possible number of subpiles based on the power of two (that is, eight tokens must become one subpile of eight, not two subpiles of four).

To evaluate a position, tabulate the number of each size subpile—if there is an even number of each size subpile (for example, two subpiles of eight, two subpiles of two, and four subpiles of one), the position is balanced. If there is an odd number of any size subpile (for example, two subpiles of eight, one subpile of four, and three subpiles of two), the position is unbalanced.

For example, if the starting position consists of three heaps, with three, five, and seven tokens, respectively (a 3-5-7 position), as in figure 2a below, each heap is broken into subpiles (figure 2b) based on powers of two (figure 2c), and the number of each size subpile is tabulated (figure 2d) to evaluate the position (which, in this case, is unbalanced).

XXX	XX X	2 (2^1), 1 (2^0)	2^0 - 3
XXXXX	XXXX X	4 (2^2), 1 (2^0)	2^1 - 2
XXXXXXX	XXXX XX X	4 (2^2), 2 (2^1), 1 (2^0)	2^2 - 2
figure 2a	figure 2b	figure 2c	figure 2d

To balance the position, the player must remove one occurrence of 2^0. This single token subpile is taken (from any pile where it occurs) to balance the position (see figure 3 below).

XXX	XX X	2 (2^1), 1 (2^0)	2^0 - **2**
XXXXX	XXXX ~~**X**~~	4 (2^2), ~~**1 (2^0)**~~	2^1 - 2
XXXXXXX	XXXX XX X	4 (2^2), 2 (2^1), 1 (2^0)	2^2 - 2
figure 3a	figure 3b	figure 3c	figure 3d

Common Nim (where the loser is forced to take the last token) employs an alternate end game (see section 4).

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4 *An unbalanced position is also referred to in Nim literature as an "uneven position," a "safe position," or a "winning position."*

3.2 *Complement Method.* The complement method is used as an optimal strategy for single-pile Nim (see figure 4 below, as played with n=5). The single heap (figure 4a) is divided into subpiles: a subpile of one (common Nim only), as many subpiles of n+1 as possible, and a subpile containing the remaining tokens (figure 4b). (If there are no remaining tokens—that is, the total number of tokens is equal to a multiple of n+1, plus one—the position is balanced, and the first player must move randomly and count on his opponent to make an error for a chance to employ this strategy for a win.) The first player (employing the optimal strategy) takes the remainder (figure 4c), leaving his opponent with a balanced position. The second player takes x tokens from the heap (figure 4d). The first player then takes the complement—that is, (n+1) – x tokens—from the heap (figure 4e). The game continues in this way (figures 4f and 4g) until, in common Nim, the second player is forced to take the remaining token, the subpile of one (figure 4h), or in straight Nim, the last complement is taken.

	Common Nim	Straight Nim
figure 4a	XXXXXXXXXXXXXXX (15 tokens)	XXXXXXXXXXXXXXX (15 tokens)
figure 4b	X XXXXXX XXXXXX XX	XXXXXX XXXXXX XXX
figure 4c	X XXXXXX XXXXXX ~~XX~~	XXXXXX XXXXXX ~~XXX~~
figure 4d	X XXXXXX X ~~XXXXX~~ ~~XX~~	XXXXXX X ~~XXXXX~~ ~~XXX~~
figure 4e	X XXXXXX ~~X~~ ~~XXXXX~~ ~~XX~~	XXXXXX ~~X~~ ~~XXXXX~~ ~~XXX~~
figure 4f	X XXXX ~~XX~~ ~~X~~ ~~XXXXX~~ ~~XX~~	XXXX ~~XX~~ ~~X~~ ~~XXXXX~~ ~~XXX~~
figure 4g	X ~~XXXX~~ ~~XX~~ ~~X~~ ~~XXXXX~~ ~~XX~~	~~XXXX~~ ~~XX~~ ~~X~~ ~~XXXXX~~ ~~XXX~~
figure 4h	~~X~~ ~~XXX~~ ~~XXX~~ ~~X~~ ~~XXXXX~~ ~~XX~~	~~XXX~~ ~~XXX~~ ~~X~~ ~~XXXXX~~ ~~XXX~~

3.3 *Binary Numbers.* As in the subpile method, binary numbers are used to evaluate whether a particular position is balanced or unbalanced, and if the position is unbalanced, what move can be made to balance the position for the opponent's play. To employ binary numbers in an optimal strategy, the first step is to render the number of tokens in each nim heap as a binary number. For example, if the position is 3-5-7 (as in figure 5a), write the number of tokens in each heap in binary (figure 5b). (All binary numbers should have the same number of digits; add 0s to the left of smaller numbers, if necessary, as in the first row of the example.)

To determine if the position is balanced, add the numbers in columns (using base 10), but without carrying numbers (figure 5c). The total is referred to as the nim sum. If the total of each column is an even number, the position is balanced. If the total of any one column is an odd number, the position is unbalanced. (In the example, because the first column [as read from the right] is an odd number, 3, the position is unbalanced.)

XXX	011	011
XXXXX	101	101
XXXXXXX	111	<u>111</u>
		223
figure 5a	figure 5b	figure 5c

The object is to balance the position for your opponent's turn. (If the position is balanced on your turn, you must count on your opponent to make an error in play, and leave an unbalanced position, before an optimal strategy may be employed.)

In the example above, the nim sum is 223; to balance this position, the nim sum must be changed to 222 by removing one token from the first (right-most) column. Each of the rows here has a token in the first column; therefore, one token may be removed from any heap to balance the position.

Using this process, you may offer your opponent a balanced position at each turn, forcing him to unbalance the position to your advantage in either straight or common Nim.

A nim sum may also be calculated using binary numbers through an iterative similar/dissimilar (S/D) operation, where each heap is compared to the next (see figure 6). For example, in comparing the first two rows, both contain a "1" in the first (read from the right) column. These similar digits are represented as a 0. In the second and third columns, one row contains a "0," while the other contains a "1." These dissimilar digits are represented as a 1. Therefore, the comparison of the first two rows, using the S/D operation, yields a sum of 110. Comparing the third row with the sum of the first two rows, in the first column, there is a "1" and a "0," which are dissimilar, and in the second and third columns, both numbers are "1," which are similar. Therefore, the nim sum, as calculated using the S/D operation, is 001, which tells us that the total position is unbalanced, and that one token must be removed from the first column to balance the position.

XXX	011
XXXXX	<u>101</u>
S/D composite of first two rows:	**110**
XXXXXXX	<u>111</u>
S/D composite of entire position:	**001**

figure 6

A Nim-playing machine can be built (as described in Fuchs, 1971) that uses the S/D operation on binary numbers to calculate winning positions.

3.4 *Nimbers.* A nimber addition table, such as the one in figure 7, can help to calculate a balanced position. In any configuration with three heaps, one would look up the intersection of the number of tokens in one heap with the number of tokens in another heap to find out how many tokens should be in the third heap to create a balanced position. For example, in a 3-5-7 configuration, the intersection of 3 and 5 is 6. Therefore, if the heap of 7 is reduced to a heap of 6, the configuration will be balanced. In addition, the intersection of 3 and 7 is 4, and the intersection of 5 and 7 is 2. Therefore, according to the nimber addition table, removing one token from any heap will balance a 3-5-7 configuration.

But if the configuration is 3-4-5, the intersection of 3 and 4 yields 7. Tokens cannot be added to a heap; therefore, two different numbers must be checked—for example, 4 and 5. The intersection of 4 and 5 is 1, so to balance a 3-4-5 configuration, two tokens should be removed from the first heap.

Nimbers make use of an algebraic equation, *a+*b+*n=*0, where *0 represents a balanced position, *a and *b represent the nimbers of tokens in two of the heaps, and *n represents the nimber of tokens in the third heap. It is important to remember that these are nimbers, not numbers (using these as numbers would yield n=-9 for a=4 and b=5).

Nimbers need not be used when the field of play is reduced to two heaps because the only balanced position is (x,x), where both heaps have the same number of tokens. A three-dimensional nimber addition table, although unwieldy, could be used to calculate balance positions for a game with four heaps.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14	17	16	19	18	21
2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13	18	19	16	17	22
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12	19	18	17	16	23
4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11	20	21	22	23	16
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10	21	20	23	22	17
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9	22	23	20	21	18
7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8	23	22	21	20	19
8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7	24	25	26	27	28
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6	25	24	27	26	29
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5	26	27	24	25	30
11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4	27	26	25	24	31
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3	28	29	30	31	24
13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2	29	28	31	30	25
14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1	30	31	28	29	26
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0	31	30	29	28	27
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	0	1	2	3	4
17	16	19	18	21	20	23	22	25	24	27	26	29	28	31	30	1	0	3	2	5
18	19	16	17	22	23	20	21	26	27	24	25	30	31	28	29	2	3	0	1	6
19	18	17	16	23	22	21	20	27	26	25	24	31	30	29	28	3	2	1	0	7
20	21	22	23	16	17	18	19	28	29	30	31	24	25	26	27	4	5	6	7	0

figure 7. nimber addition table

4.0 NIM END GAMES

For common Nim, the unbalanced to balanced position strategy may need to be modified in the end game. An end game occurs when a balancing move would reduce the field of play to an even number of one-token heaps.

For example, figure 8 shows a situation in which a player must recognize that a win can be achieved in common Nim by removing all three counters in the first heap, rather than balancing the position by removing only two of the counters from the first heap (figure $8b_1$), as one would to win in straight Nim (figure $8b_2$). If a player were to reduce the field to two one-token heaps while playing common Nim, his opponent would win by removing one of the two heaps, leaving the lone token in the remaining heap.

XXX
X
figure 8a

~~XXX~~	X~~XX~~
X	X
figure $8b_1$ (Common)	figure $8b_2$ (Straight)

5.0 REFERENCES

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